MATHEMATICS 9th Science Group

7/18/2020

Chapter 4.

ALGEBRAIC EXPRESSIONS AND ALGEBRAIC FORMULAS

Contents

Algebraic Expressions

Algebra is a generalization of arithmetic. Recall that when operations of addition and subtraction are applied to algebraic terms, we obtain an algebraic expression. For instance, $5x^2 - 3x + \frac{2}{\sqrt{x}}$, $3xy + \frac{3}{x}\,(x \neq 0)$ are algebraic expressions.

Polynomials

it is a polynomial A polynomial in the variable x is an algebraic expression of the form

$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0, \quad a_n \neq 0$ ...... (i)

where n, the highest power of x, is a non-negative integer called the degree of the polynomial and each coefficient a_n, is a real number. The coefficient a_n of the highest power of x is called the *leading coefficient* of the polynomial. $2x^4y^2 + x^2y^2 + 8x$ is a polynomial in two variables x and y having degree 6 (4+2=6).

Rational Expression

The quotient $\frac{p(x)}{q(x)}$ of two polynomials, $p(x)$ and $q(x)$, where $q(x)$ is a non-zero polynomial, is called a rational expression.

For example, $\frac{2x+5}{5x-1}$, $5x- \neq 0$ is a rational expression.

Note:

Every polynomial $p(x)$ can be regarded as a rational expression, since we can write $p(x)$ as $\frac{p(x)}{1}$.

Thus, every polynomial is a rational expression, but every rational expression need not be a polynomial.

Algebraic formulas

(i). $(a+b)^2 = a^2 + b^2 + 2ab$

(ii). $(a-b)^2 = a^2 + b^2 - 2ab$

(iii). $x^2 - y^2 = (x-y)(x+y)$

(iv) $(x+y)^3 = x^3 + y^3 + 3xy(x+y)$

(v) $(x-y)^3 = x^3 - y^3 - 3xy(x-y)$

(vi) $\left(x+\frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x+\frac{1}{x}\right)$

(vii) $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

(viii) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

EXERCISE4.1

Q#1) Identify whether the following algebraic expressions are polynomials (Yes or No).

(i). $3x^2 + \frac{1}{x} - 5$

Sol: No, it is not a polynomial because it contains the term $\frac{1}{x}$.

(ii). $3x^3 - 4x^2 - x\sqrt{x} + 3$

Sol: No, it is not a polynomial because it contains the term $x\sqrt{x}$.

(iii). $x^2 - 3x + \sqrt{2}$

Sol: Yes, it is a polynomial because all powers are non-negative integers.

(iv). $\frac{3x}{2x-1} + 8$

Sol: No, it is not a polynomial because it contains the term $\frac{3x}{2x-1}$.

Q#2) State whether each of the following expressions is a rational expression or not.

(i). $\frac{3\sqrt{x}}{3\sqrt{x}+5}$

Sol: It is not rational expression.

(ii). $\frac{x^3-2x^2+\sqrt{3}}{2+3x-x^2}$

Sol: It is not rational expression.

(iii). $\frac{x^2+6x+9}{x^2-9}$

Sol: It a rational expression.

(iv). $\frac{2\sqrt{x}+3}{2\sqrt{x}-3}$

Sol: It is not rational expression.

Q#3) Reduce the following rational expressions to the lowest form.

(i). $\frac{120\,x^2\,y^3\,z^5}{30\,x^3 y\,z^2}$

Sol: $\frac{120\,x^2\,y^3\,z^5}{30\,x^3 y\,z^2} = 4x^{2-3}\,y^{3-1}\,z^{5-2}$

$= 4x^{-1}\,y^2\,z^3$

$= \frac{4\,y^2\,z^3}{x}$

(ii). $\frac{8a\,(x+1)}{2(x^2-1)}$

Sol: $\frac{8a\,(x+1)}{2(x^2-1)} = \frac{4a\,(x+1)}{(x-1)(x+1)}$

$= \frac{4a}{(x-1)}$

$= \frac{4\,a}{x-1}$

(iii). $\frac{(x+y)^2-4xy}{(x-y)^2}$

Sol: $\frac{(x+y)^2-4xy}{(x-y)^2} = \frac{x^2+y^2+2xy-4xy}{x^2+y^2-2xy}$

$== \frac{x^2+y^2-2xy}{x^2+y^2-2xy} = 1$

(iv). $\frac{(x^3-y^3)(x^2-2xy+y^2)}{(x-y)(x^2+xy+y^2)}$

Sol: $\frac{(x^3-y^3)(x^2-2xy+y^2)}{(x-y)(x^2+xy+y^2)} = \frac{(x^3-y^3)(x^2-2xy+y^2)}{(x^3-y^3)}$

$= (x-y)^2$

(v). $\frac{(x+2)(x^2-1)}{(x+1)(x^2-4)}$

Sol: $\frac{(x+2)(x^2-1)}{(x+1)(x^2-4)} = \frac{(x+2)(x-1)(x+1)}{(x+1)(x^2-2^2)}$

$= \frac{(x+2)(x-1)}{(x-2)(x+2)} = \frac{x-1}{x-2}$

(vi). $\frac{x^2-4x+4}{2x^2-8}$

Sol: $\frac{x^2-4x+4}{2x^2-8} = \frac{x^2-2(x)(2)+(2)^2}{2(x^2-4)}$

$$= \frac{(x-2)^2}{2(x^2-2^2)}$$
$$= \frac{(x-2)^2}{2(x-2)(x+2)}$$
$$= \frac{x-2}{2(x+2)}$$

(vii). $\frac{64x^5-64x}{(8x^2+8)(2x+2)}$

Sol: $\frac{64x^5-64x}{(8x^2+8)(2x+2)} = \frac{64x(x^4-1)}{8(x^2+1)2(x+1)}$
$$= \frac{4x((x^2)^2-(1)^2)}{(x^2+1)(x+1)}$$
$$= \frac{4x(x^2+1)(x^2-1)}{(x^2+1)(x+1)}$$
$$= \frac{4x(x-1)(x+1)}{(x+1)}$$
$$= 4x(x-1)$$

(viii). $\frac{9x^2-(x^2-4)^2}{4+3x-x^2}$

Sol: $\frac{9x^2-(x^2-4)^2}{4+3x-x^2} = \frac{(3x)^2-(x^2-4)^2}{4+3x-x^2}$
$$= \frac{(3x-(x^2-4))(3x+(x^2-4))}{4+3x-x^2}$$
$$= \frac{(3x-x^2+4)(3x+x^2-4)}{4+3x-x^2}$$
$$= 3x+x^2-4$$

Q#4) Evaluate (a). $\frac{x^3y-2z}{xz}$ **for**

(i). $x = 3, y = -1\ and\ z = -2$

Sol: As given $\frac{x^3y-2z}{xz}$

Putt $x = 3, y = -1\ and\ z = -2$ **in above**

$$\frac{x^3y-2z}{xz} = \frac{(3)^3(-1)-2(-2)}{(3)(-2)}$$
$$= \frac{(27)(-1)+4}{-6}$$
$$= \frac{-27+4}{-6}$$
$$= \frac{-23}{-6} = \frac{23}{6} = 3\frac{5}{6}$$

(ii). $x = -1, y = -9\ and\ z = 4$

Sol: As given $\frac{x^3y-2z}{xz}$

Putt $x = -1, y = -9\ and\ z = 4$ **in above**

$$\frac{x^3y-2z}{xz} = \frac{(-1)^3(-9)-2(4)}{(-1)(4)}$$
$$= \frac{(-1)(-9)-8}{-4}$$
$$= \frac{+9-8}{-4}$$
$$= \frac{1}{-4} = -\frac{1}{4}$$

(b). $\frac{x^2y^3-5z^4}{xyz}$ **for** $x = 4, y = -2\ and\ z = -1$

Sol: As given $\frac{x^2y^3-5z^4}{xyz}$

Putt $x = 4, y = -2\ and\ z = -1$ **in above**

$$\frac{x^2y^3-5z^4}{xyz} = \frac{(4)^2(-2)^3-5(-1)^4}{(4)(-2)(-1)}$$
$$= \frac{(16)(-8)-5(1)}{8}$$
$$= \frac{-128-5}{8}$$
$$= \frac{-133}{8} = -16\frac{5}{8}$$

Q#5) Perform the indicated operation and simplify.

(i). $\frac{15}{2x-3y} - \frac{4}{3y-2x}$

Sol: $\frac{15}{2x-3y} - \frac{4}{3y-2x} = \frac{15}{2x-3y} - \frac{4}{-(2x-3y)}$
$$= \frac{15}{2x-3y} + \frac{4}{2x-3y}$$
$$= \frac{15+4}{2x-3y}$$
$$= \frac{19}{2x-3y}$$

(ii). $\frac{1+2x}{1-2x} - \frac{1-2x}{1+2x}$

Sol: $\frac{1+2x}{1-2x} - \frac{1-2x}{1+2x} = \frac{(1+2x)^2-(1-2x)^2}{(1-2x)(1+2x)}$
$$= \frac{[(1)^2+(2x)^2+2(1)(2x)]-[(1)^2+(2x)^2-2(1)(2x)]}{(1)^2-(2x)^2}$$
$$= \frac{1+4x^2+4x-1-4x^2+4x}{1-4x^2}$$
$$= \frac{8}{1-4x^2}$$

(iii). $\frac{x^2-25}{x^2-36} - \frac{x+5}{x+6}$

Sol $\frac{x^2-25}{x^2-36} - \frac{x+5}{x+6} = \frac{x^2-5^2}{x^2-6^2} - \frac{x+5}{x+6}$
$$= \frac{(x+5)(x-5)}{(x+6)(x-6)} - \frac{x+5}{x+6}$$
$$= \frac{(x+5)(x-5)-(x-6)(x+5)}{(x+6)(x-6)}$$
$$= \frac{(x^2-25)-(x^2+5x-6x-30)}{x^2-36}$$
$$= \frac{(x^2-25)-(x^2-x-30)}{x^2-36}$$
$$= \frac{x^2-25-x^2+x+30}{x^2-36}$$
$$= \frac{x+5}{x^2-36}$$

(iv). $\frac{x}{x-y} - \frac{y}{x+y} - \frac{2xy}{x^2-y^2}$

Sol: $\frac{x}{x-y} - \frac{y}{x+y} - \frac{2xy}{x^2-y^2} = \frac{x}{x-y} - \frac{y}{x+y} - \frac{2xy}{(x-y)(x+y)}$
$$= \frac{x(x+y)-y(x-y)-2xy}{(x-y)(x+y)}$$
$$= \frac{x^2+xy-xy+y^2-2xy}{(x-y)(x+y)}$$
$$= \frac{(x-y)^2}{(x-y)(x+y)}$$
$$= \frac{x-y}{x+y}$$

(v). $\frac{x-2}{x^2+6x+9} - \frac{x+2}{2x^2-18}$

Sol: $\frac{x-2}{x^2+6x+9}-\frac{x+2}{2x^2-18}=\frac{x-2}{(x)^2+2(x)(3)+(3^2)}-\frac{x+2}{2(x^2-9)}$

$$=\frac{x-2}{(x+3)^2}-\frac{x+2}{2(x^2-3^2)}$$

$$=\frac{x-2}{(x+3)^2}-\frac{x+2}{2(x-3)(x+3)}$$

$$=\frac{2(x-2)(x-3)-(x+2)(x+3)}{2(x-3)(x+3)^2}$$

$$=\frac{2(x^2-3x-2x+6)-(x^2+3x+2x+6)}{2(x-3)(x+3)^2}$$

$$=\frac{2(x^2-5x+6)-(x^2+5x+6)}{2(x-3)(x+3)^2}$$

$$=\frac{2x^2-10x+12-x^2-5x-6}{2(x-3)(x+3)^2}$$

$$=\frac{x^2-15x+6}{2(x-3)(x+3)^2}$$

(iv). $\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{x^2+1}-\frac{4}{x^4-1}$

Sol: $\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{x^2+1}-\frac{4}{x^4-1}$

$$=\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{x^2+1}-\frac{4}{(x-1)(x+1)(x^2+1)}$$

$$=\frac{1(x+1)(x^2+1)-1(x-1)(x^2+1)-2(x+1)(x-1)-4}{(x-1)(x+1)(x^2+1)}$$

$$=\frac{(x^3+x+x^2+1)-(x^3+x-x^2-1)-2(x^2-1)-4}{x^4-1}$$

$$=\frac{x^3+x+x^2+1-x^3-x+x^2+1-2x^2+2-4}{x^4-1}$$

$$=\frac{2x^2+4-2x^2-4}{x^4-1}$$

$$=\frac{0}{x^4-1}=0$$

Q#6) Perform the indicated operation and simplify.

(i). $(x^2-49).\frac{5x+2}{x+7}$

Sol: $(x^2-49).\frac{5x+2}{x+7}=(x^2-7^2).\frac{5x+2}{x+7}$

$$=(x-7)(x+7).\frac{5x+2}{x+7}$$

$$=(x-7)(5x+2)$$

(ii). $\frac{4x-12}{x^2-9}\div\frac{18-2x^2}{x^2+6x+9}$

Sol: $\frac{4x-12}{x^2-9}\div\frac{18-2x^2}{x^2+6x+9}=\frac{4(x-3)}{x^2-3^2}\times\frac{(x)^2+2(x)(3)+(3^2)}{2(9-x^2)}$

$$=\frac{4(x-3)}{(x-3)(x+3)}\times\frac{(x+3)^2}{2(3-x)(3+x)}$$

$$=\frac{2}{1}\times\frac{1}{(3-x)}=\frac{2}{3-x}$$

(iii). $\frac{x^6-y^6}{x^2-y^2}\div(x^4+x^2y^2+y^4)$

Sol: $\frac{x^6-y^6}{x^2-y^2}\div(x^4+x^2y^2+y^4)$

$$=\frac{(x^2)^3-(y^2)^3}{(x-y)(x+y)}\times\frac{1}{(x^4+x^2y^2+y^4)}$$

$$=\frac{(x^2-y^2)((x^2)^2+(x^2)(y^2)+(y^2)^2)}{(x-y)(x+y)}\times\frac{1}{(x^4+x^2y^2+y^4)}$$

$$=\frac{(x-y)(x+y)(+x^2y^2+y^4)}{(x-y)(x+y)}\times\frac{1}{(x^4+x^2y^2+y^4)}$$

$$=1$$

(iv). $\frac{x^2-1}{x^2+2x+1}\cdot\frac{x+5}{1-x}$

Sol: $\frac{x^2-1}{x^2+2x+1}\cdot\frac{x+5}{1-x}=\frac{(x-1)(x+1)}{(x+1)^2}\cdot\frac{x+5}{1-x}$

$$=\frac{(x-1)}{(x+1)}\cdot\frac{x+5}{-(x-1)}$$

$$=-\frac{(x+5)}{(x+1)}$$

(v). $\frac{x^2+xy}{y(x+y)}\cdot\frac{x^2+xy}{y(x+y)}\div\frac{x^2-x}{xy-2y}$

Sol: $\frac{x^2+xy}{y(x+y)}\cdot\frac{y^2+xy}{y(x+y)}\div\frac{x^2-x}{xy-2y}$

$$=\frac{x(x+y)}{y(x+y)}\cdot\frac{x(x+y)}{y(x+y)}\times\frac{xy-2y}{x^2-x}$$

$$=\frac{x}{y}\cdot\frac{x}{y}\times\frac{y(x-2)}{x(x-1)}$$

$$=\frac{x}{y}\times\frac{(x-2)}{(x-1)}$$

$$=\frac{x(x-2)}{y(x-1)}$$

Algebraic formulas

(i). $(a+b)^2+(a-b)^2=2(a^2+b^2)$

(ii). $(a+b)^2-(a-b)^2=4ab$

(iii). $(a+b+c)^2=a^2+b^2+c^2+2(ab+ac+ca)$

(iv) $(x+y)^3=x^3+y^3+3xy(x+y)$

(v) $(x-y)^3=x^3-y^3-3xy(x-y)$

(vi) $\left(x+\frac{1}{x}\right)^3=x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)$

(vii) $x^3+y^3=(x+y)(x^2-xy+y^2)$

(viii) $x^3-y^3=(x-y)(x^2+xy+y^2)$

EXERCISE 4.2

Q#1).

(i) If $a+b=10$ and $a-b=6$, then find the value of a^2+b^2

Solution: As given $a+b=10$ and $a-b=6$

We find a^2+b^2=?

Using the identity

$$(a+b)^2+(a-b)^2=2(a^2+b^2)$$

Put values

$$(10)^2+(6)^2=2(a^2+b^2)$$

$$100+36=2(a^2+b^2)$$

$$136=2(a^2+b^2)$$

$$a^2+b^2=\frac{136}{2}$$

$$a^2+b^2=68$$

Which is required.

(ii) If $a+b=5$ and $a-b=\sqrt{17}$, then find the value of ab

Solution: As given $a+b=5$ and $a-b=\sqrt{17}$

We find ab=?

Using the identity

$$(a+b)^2-(a-b)^2=4ab$$

Put values

$$(5)^2-(\sqrt{17})^2=4ab$$
$$25-17=4ab$$
$$8=4ab$$
$$ab=\frac{8}{4}$$
$$ab=2$$

Which is required.

Q#2) If $a^2+b^2+c^2=45$ and $a+b+c=-1$, then find the value of $ab+bc+ca$

Solution: As given $a^2+b^2+c^2=45$ and $a+b+c=-1$

We find $ab+bc+ca$=?

Using the identity

$(a+b+c)^2=a^2+b^2+c^2+2(ab+ac+ca)$

Put values

$$(-1)^2=45+2(ab+bc+ca)$$
$$1=45+2(ab+bc+ca)$$
$$1-45=2(ab+bc+ca)$$
$$-44=2(ab+bc+ca)$$
$$ab+bc+ca=-\frac{44}{2}$$
$$ab+bc+ca=-22$$

Which is required.

Q#3) If $m+n+p=10$ and $mn+np+mp=27$, then find the value of $m^2+n^2+p^2$

Solution: As given $m+n+p=10$ and $mn+np+mp=27$

We find $m^2+n^2+p^2$=?

Using the identity

$(m+n+p)^2=m^2+n^2+p^2+2(mn+np+mp)$

Put values

$$(10)^2=m^2+n^2+p^2+2(27)$$
$$100=m^2+n^2+p^2+54$$
$$100-54=m^2+n^2+p^2$$
$$m^2+n^2+p^2=46$$

Which is required.

Q#4) If $x^2+y^2+z^2=78$ and $xy+yz+zx=59$, then find the value of $x+y+z$

Solution: As given $x^2+y^2+z^2=78$ and $xy+yz+zx=59$

We find $x+y+z$ =?

Using the identity

$(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)$

Put values

$$(x+y+z)^2=78+2(59)$$
$$(x+y+z)^2=78+118$$
$$(x+y+z)^2=196$$

On taking square root, we get

$$\sqrt{(x+y+z)^2}=\pm\sqrt{196}$$
$$x+y+z=\pm 14$$

Which is required.

Q#5) If $x^2+y^2+z^2=78$ and $xy+yz+zx=59$, then find the value of $x+y+z$

Solution: As given $x+y+z=12$ and $x^2+y^2+z^2=64$

We find $xy+yz+zx$ =?

Using the identity

$(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)$

Put values

$$(12)^2=64+2(xy+yz+zx)$$
$$144=64+2(xy+yz+zx)$$
$$144-64=2(xy+yz+zx)$$
$$80=2(xy+yz+zx)$$
$$\frac{80}{2}=(xy+yz+zx)$$
$$xy+yz+zx=40$$

Which is required.

Q#6) If $x+y=7$ and $xy=12$, then find the value of x^3+y^3

Solution: As given $x+y=7$ and $xy=12$

We find x^3+y^3 =?

Using the identity

$$(x+y)^3=x^3+y^3+3xy(x+y)$$

Put values

$$(7)^3=x^3+y^3+3(12)(7)$$
$$343=x^3+y^3+252$$
$$343-252=x^3+y^3$$
$$x^3+y^3=91$$

Which is required.

Q#7) If $3x+4y=11$ and $xy=12$, then find the value of $27x^3+64y^3$

Solution: As given $3x+4y=11$ and $xy=12$

We find $27x^3+64y^3$ =?

Using the identity

$$(x+y)^3=x^3+y^3+3xy(x+y)$$

It becomes

$$(3x+4y)^3=(3x)^3+(4y)^3+3(3x)(4y)(3x+4y)$$
$$(3x+4y)^3=27x^3+64y^3+36(xy)(3x+4y)$$
$$(11)^3=27x^3+64y^3+36(12)(11)$$
$$1331=27x^3+64y^3+4752$$
$$1331-4752=27x^3+64y^3$$
$$27x^3+64y^3=-3421$$

Q#8) If $x-y=4$ and $xy=21$, then find the value of x^3-y^3

Solution: As given $x-y=4$ and $xy=21$

We find x^3-y^3 =?

Using the identity

$(x-y)^3=x^3-y^3-3xy(x-y)$

Put values

$$(4)^3=x^3-y^3-3(21)(4)$$
$$64=x^3+y^3-252$$
$$64+252=x^3+y^3$$
$$x^3+y^3=316$$

Which is required value.

Q#9) If $5x-6y=13$ and $xy=6$, then find the value of $125x^3-216y^3$

Solution: As given $5x-6y=13$ and $xy=6$

We find $27x^3 + 64y^3 = ?$
Using the identity
$$(x+y)^3 = x^3 + y^3 + 3xy(x+y)$$
It becomes
$$(5x-6y)^3 = (5x)^3 - (6y)^3 - 3(5x)(6y)(5x - 6y)$$
$$(5x-6y)^3 = 125x^3 - 216y^3 - 90(xy)(5x-6y)$$
$$(13)^3 = 125x^3 - 216y^3 - 90(6)(13)$$
$$2197 = 125x^3 - 216y^3 - 7020$$
$$2197 + 7020 = 125x^3 - 216y^3$$
$$125x^3 - 216y^3 = 9217$$
Which is required value.

Q#10) If $x + \frac{1}{x} = 3$ then find the value of $x^3 + \frac{1}{x^3}$

Solution: As given $x + \frac{1}{x} = 3$

We find $x^3 + \frac{1}{x^3} = ?$

Using the identity
$$\left(x+\frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x+\frac{1}{x}\right)$$
Put values
$$(3)^3 = x^3 + \frac{1}{x^3} + 3(3)$$
$$27 = x^3 + \frac{1}{x^3} + 9$$
$$27 - 9 = x^3 + \frac{1}{x^3}$$
$$x^3 + \frac{1}{x^3} = 18$$
Which is required value.

Q#11) If $x - \frac{1}{x} = 7$ then find the value of $x^3 - \frac{1}{x^3}$

Sol: As given $x - \frac{1}{x} = 7$

We find $x^3 - \frac{1}{x^3} = ?$

Using the identity
$$\left(x-\frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3\left(x-\frac{1}{x}\right)$$
Put values
$$(7)^3 = x^3 - \frac{1}{x^3} - 3(7)$$
$$343 = x^3 - \frac{1}{x^3} - 21$$
$$343 + 21 = x^3 - \frac{1}{x^3}$$
$$x^3 - \frac{1}{x^3} = 364$$

Q#12) If $(3x + \frac{1}{3x}) = 5$ then find the value of $27x^3 + \frac{1}{27x^3}$

Solution: As given$3x + \frac{1}{3x} = 5$

We find $27x^3 + \frac{1}{27x^3} = ?$

Using the identity
$$\left(x+\frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x+\frac{1}{x}\right)$$
It becomes
$$\left(3x+\frac{1}{3x}\right)^3 = (3x)^3 + \frac{1}{(3x)^3} + \left(3x+\frac{1}{3x}\right)$$
$$\left(3x+\frac{1}{3x}\right)^3 = 27x^3 + \frac{1}{27x^3} + \left(3x+\frac{1}{3x}\right)$$
$$(5)^3 = 27x^3 + \frac{1}{27x^3} + 3(5)$$
$$125 = 27x^3 + \frac{1}{27x^3} + 15$$
$$125 - 15 = 27x^3 + \frac{1}{27x^3}$$
$$27x^3 + \frac{1}{27x^3} = 110$$

Q#13) If $(5x - \frac{1}{5x}) = 6$ then find the value of $125x^3 - \frac{1}{125x^3}$

Solution: As given $5x - \frac{1}{5x} = 6$

We find $125x^3 - \frac{1}{125x^3} = ?$

Using the identity
$$\left(x-\frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3\left(x-\frac{1}{x}\right)$$
It becomes
$$\left(5x-\frac{1}{5x}\right)^3 = (3x)^3 - \frac{1}{(3x)^3} - \left(5x-\frac{1}{5x}\right)$$
$$\left(5x-\frac{1}{5x}\right)^3 = 125x^3 - \frac{1}{125x^3} + \left(5x-\frac{1}{5x}\right)$$
$$(6)^3 = 125x^3 - \frac{1}{125x^3} - 3(6)$$
$$216 = 125x^3 - \frac{1}{125x^3} - 18$$
$$216 + 18 = 125x^3 - \frac{1}{125x^3}$$
$$125x^3 - \frac{1}{125x^3} = 234$$

Q#15)

(i). $x^3 - y^3 - x + y$

Sol: $x^3 - y^3 - x + y$

$= (x-y)(x^2+xy+y^2) - (x-y)$

$= (x-y)(x^2+xy+y^2-1)$

(ii). $8x^3 - \frac{1}{27y^3}$

Sol: $8x^3 - \frac{1}{27y^3}$

$= (2x)^3 - \left(\frac{1}{3y}\right)^3$

$= \left(2x - \frac{1}{3y}\right)\left((2x)^2 + (2x)\left(\frac{1}{3y}\right) + \left(\frac{1}{3y}\right)^2\right)$

$= \left(2x - \frac{1}{3y}\right)\left(4x^2 + \frac{2x}{3y} + \frac{1}{9y^2}\right)$

Q#16) Find the product, using formulas.

(i). $(x^2+y^2)(x^4 - x^2y^2 + y^4)$

Solution: $(x^2+y^2)(x^4 - x^2y^2 + y^4)$

$= (x^2+y^2)[(x^2)^2 - (x^2)(y^2) + (y^2)^2]$

Using identity

$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$= (x^2)^3 + (y^2)^3$

$= x^6 + y^6$

(ii). $(x^3-y^3)(x^6+x^3y^3+y^6)$

Sol: $(x^3-y^3)(x^6+x^3y^3+y^6)$

$= (x^3-y^3)[(x^3)^2+(x^3)(y^3)+(y^3)^2]$

Using identity

$x^3-y^3=(x-y)(x^2+xy+y^2)$

$=(x^3)^3-(y^3)^3$

$=x^9-y^9$

(iii). $(x-y)(x+y)(x^2+y^2)(x^2+xy+y^2)(x^2-xy+y^2)(x^4-x^2y^2+y^4)$

Sol: $(x-y)(x+y)(x^2+y^2)(x^2+xy+y^2)(x^2-xy+y^2)(x^4-x^2y^2+y^4)$

$=[(x-y)(x^2+xy+y^2)][(x+y)(x^2-xy+y^2)][(x^2+y^2)((x^2)^2-(x^2)(y^2)+(y^2)^2)]$

Using identity

$x^3+y^3=(x+y)(x^2-xy+y^2)$

$=[x^3-y^3][x^3+y^3][(x^2)^3+(y^2)^3]$

$=[(x^3)^2-(y^3)^2][x^6+y^6]$

$=(x^6-y^6)(x^6+y^6)$

$=(x^6)^2-(y^6)^2$

$=x^{12}-y^{12}$

(iv). $(2x^2-1)(2x^2+1)(4x^4+2x^2+1)(4x^4-2x^2+1)$

Sol: $(2x^2-1)(2x^2+1)(4x^4+2x^2+1)(4x^4-2x^2+1)$

$=[(2x^2-1)((2x^2)^2+(2x^2)(1)+(1)^2][(2x^2+1)((2x^2)^2-(2x^2)(1)+(1)^2]$

$=[(2x^2)^3-(1)^3][(2x^2)^3+(1)^3]$

$=(8x^6-1)(8x^6+1)$

$=(8x^6)^2-(1)^2$

$=64x^{12}-1$

Surd:

An irrational radical with rational radicand is called a surd.

That is $\sqrt[n]{a}$ surd if a is rational and $\sqrt[n]{a}$ is irrational.

For example, $\sqrt{2}, \sqrt[4]{5}, \sqrt{10}$

Also, $\sqrt{\pi}$ is not a surd because π is not rational.

$\sqrt{10+\sqrt{2}}$ is not a surd because $10+\sqrt{2}$ is not a rational number.

EXERCISE 4.3

1. Express each of the following surd in the simplest foam.

(i) $\sqrt{180}$

Solution: $\sqrt{180}=\sqrt{2\times2\times3\times3\times5}$

$=\sqrt{2^2\times3^2\times5}$

$=2\times3\sqrt{5}$

$=6\sqrt{5}$

(ii) $3\sqrt{162}$

Solution: $3\sqrt{162}=3\sqrt{2\times3\times3\times3\times3}$

$=3\sqrt{2\times3^2\times3^2}$

$=3\times3\times3\sqrt{2}$

$=27\sqrt{2}$

(iii) $\frac{3}{4}\sqrt[3]{128}$

Solution: $\frac{3}{4}\sqrt[3]{128}=\frac{3}{4}\sqrt[3]{2\times2\times2\times2\times2\times2\times2}$

$=\frac{3}{4}\sqrt[3]{2^3\times2^3\times2}$

$=\frac{3}{4}(2\times2\sqrt[3]{2})$

$=\frac{3}{4}(4\sqrt[3]{2})$

$=\mathbf{3\sqrt[3]{2}}$

(iv) $\sqrt[5]{96x^6y^7z^8}$

Solution: $\sqrt[5]{96x^6y^7z^8}$

$=\sqrt[5]{2\times2\times2\times2\times2\times3\times x^5\times y^5\times z^5\times x\times y^2\times z^3}$

$=\sqrt[5]{2^5\times x^5\times y^5\times z^5\times3\times x\times y^2\times z^3}$

$=2\times x\times y\times z\sqrt[5]{3\times x\times y^2\times z^3}$

$=2xyz\sqrt[5]{3xy^2z^3}$

Q#2) Simplify

(i). $\frac{\sqrt{18}}{\sqrt{3}\sqrt{2}}$

Solution: $\frac{\sqrt{18}}{\sqrt{3}\sqrt{2}}=\frac{\sqrt{2\times3\times3}}{\sqrt{3}\sqrt{2}}=\frac{\sqrt{2}\sqrt{3}\sqrt{3}}{\sqrt{3}\sqrt{2}}=\sqrt{3}$

(ii). $\frac{\sqrt{21}\sqrt{9}}{\sqrt{63}}$

Solution: $\frac{\sqrt{21}\sqrt{9}}{\sqrt{63}}=\frac{\sqrt{3\times7}\sqrt{3\times3}}{\sqrt{3\times3\times7}}=\frac{\sqrt{3}\sqrt{7}\sqrt{3}\sqrt{3}}{\sqrt{3}\sqrt{3}\sqrt{7}}=\sqrt{3}$

(iii). $\sqrt[5]{243x^5y^{10}z^{15}}$

Solution: $\sqrt[5]{243x^5y^{10}z^{15}}$

$=\sqrt[5]{3\times3\times3\times3\times3\times x^5\times y^5\times y^5\times z^5\times z^5\times z^5}$

$=\sqrt[5]{3^5\times x^5\times y^5\times y^5\times z^5\times z^5\times z^5}$

$=(3^5\times x^5\times y^5\times y^5\times z^5\times z^5\times z^5)^{\frac{1}{5}}$

$= 3^{5\times\frac{1}{5}} \times x^{5\times\frac{1}{5}} \times y^{5\times\frac{1}{5}} \times y^{5\times\frac{1}{5}} \times z^{5\times\frac{1}{5}} \times z^{5\times\frac{1}{5}} \times z^{5\times\frac{1}{5}}$
$= 3 \times x \times y \times y \times z \times z \times z$
$= 3xy^2z^3$

(iv) $\frac{4}{5}\sqrt[3]{125}$

Solution: $\frac{4}{5}\sqrt[3]{1258} = \frac{4}{5}\sqrt[3]{5\times5\times5}$
$= \frac{4}{5}\sqrt[3]{5^3}$
$= \frac{4}{5}(5)$
$= 4$

(v). $\sqrt{21}\times\sqrt{7}\times\sqrt{3}$

Solution: $\sqrt{21}\times\sqrt{7}\times\sqrt{3} = \sqrt{7\times3}\times\sqrt{7}\times\sqrt{3}$
$= \sqrt{7}\times\sqrt{3}\times\sqrt{7}\times\sqrt{3}$
$= \left(\sqrt{7}\right)^2 \times \left(\sqrt{3}\right)^2$
$= 7\times3 = 21$

Q#3) Simplify by combining similar terms.

(i). $\mathbf{\sqrt{45} - 3\sqrt{20} + 4\sqrt{5}}$

Solution: $\sqrt{45} - 3\sqrt{20} + 4\sqrt{5} = \sqrt{3\times3\times5} - 3\sqrt{2\times2\times5} + 4\sqrt{5}$
$= \sqrt{3^2\times5} - 3\sqrt{2^2\times5} + 4\sqrt{5}$
$= 3\sqrt{5} - 3\times2\sqrt{5} + 4\sqrt{5}$
$= 3\sqrt{5} - 6\sqrt{5} + 4\sqrt{5}$
$= \sqrt{5}(3-6+4)$
$= \sqrt{5}(1)$
$= \sqrt{5}$

(ii). $4\sqrt{12} + 5\sqrt{27} - 3\sqrt{75} + \sqrt{300}$

Solution: $4\sqrt{12} + 5\sqrt{27} - 3\sqrt{75} + \sqrt{300}$
$= 4\sqrt{2\times2\times3} + 5\sqrt{3\times3\times3} - 3\sqrt{5\times5\times3} + \sqrt{2\times2\times5\times5\times3}$
$= 4\sqrt{2^2\times3} + 5\sqrt{3^2\times3} - 3\sqrt{5^2\times3} + \sqrt{2^2\times5^2\times3}$
$= 4\times2\sqrt{3} + 5\times3\sqrt{3} - 3\times5\sqrt{3} + 2\times5\sqrt{3}$
$= 8\sqrt{5} + 15\sqrt{5} - 15\sqrt{5} + 10\sqrt{3}$
$= \sqrt{3}(8+15-15+10)$
$= \sqrt{5}(18)$
$= 18\sqrt{5}$

(iii). $\sqrt{3}(2\sqrt{3}+3\sqrt{3})$

Solution: $\sqrt{3}\left(2\sqrt{3}+3\sqrt{3}\right) = \sqrt{3}(5\sqrt{3})$
$= 5\left(\sqrt{3}\right)^2 = 5(3) = 15$

(iv). $2(6\sqrt{5}-3\sqrt{5})$

Solution: $2(6\sqrt{5}-3\sqrt{5}) = 2(3\sqrt{5})$
$= 6\sqrt{5}$

Q#4) Simplify

(i). $(3+\sqrt{3})(3-\sqrt{3})$

Sol: $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)$
$= (3)^2 - \left(\sqrt{3}\right)^2$
$= 9 - 3 = 6$

(ii). $\left(\sqrt{5}+\sqrt{3}\right)^2$

Solution: $\left(\sqrt{5}+\sqrt{3}\right)^2$
$= \left(\sqrt{5}\right)^2 + \left(\sqrt{3}\right)^2 + 2(\sqrt{5})(\sqrt{3})$
$= 5 + 3 + 2\sqrt{3\times5} = 8 + 2\sqrt{15}$

(iii). $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$

Solution: $\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)$
$= \left(\sqrt{5}\right)^2 - \left(\sqrt{3}\right)^2$
$= 5 - 3 = 2$

(iv). $(\sqrt{2}+\frac{1}{\sqrt{3}})(\sqrt{2}-\frac{1}{\sqrt{3}})$

Solution: $(\sqrt{2}+\frac{1}{\sqrt{3}})(\sqrt{2}-\frac{1}{\sqrt{3}})$
$= \left(\sqrt{2}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2$
$= 2 - \frac{1}{3} = \frac{6-1}{3} = \frac{5}{3}$

(i). $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})(x+y)(x^2+y^2)$

Solution: $\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)(x+y)(x^2+y^2)$
$= \left(\left(\sqrt{x}\right)^2 - \left(\sqrt{y}\right)^2\right)(x+y)(x^2+y^2)$
$= (x-y)(x+y)(x^2+y^2)$
$= ((x)^2-(y)^2)(x^2+y^2)$
$= (x^2)^2 - (y^2)^2$
$= x^4 - y^4$

Surd:
An irrational radical with rational radicand is called a surd.
That is $\sqrt[n]{a}$ surd if a is rational and $\sqrt[n]{a}$ is irrational.
For example, $\sqrt{2}, \sqrt[4]{5}, \sqrt{10}$
Also, $\sqrt{\pi}$ is not a surd because π is not rational.
$\sqrt{10+\sqrt{2}}$ is not a surd because $10+\sqrt{2}$ is not a rational number.

Monomial surd:
A surd which contains a single term is called a monomial surd.
e.g., $\sqrt{2}, \sqrt{5}$ etc.

Binomial surd:
A surd which contains sum of two monomial surds or sum of a monomial surd and a rational number is called a binomial surd.
e.g., $\sqrt{2}+\sqrt{7}$ + or $\sqrt{12}-\sqrt{7}$ or $\sqrt{10}-\sqrt{2}$ etc.
We can extend this to the definition of a trinomial surd.

Rationalizing factor of the other
If the product of two surds is a rational number, then each surd is called the rationalizing factor of the other.

Rationalization
The process of multiplying a given surd by its rationalizing factor to get a rational number as product is called rationalization of the given surd.

Conjugate surd
Two binomial surds of second order differing only in sign connecting their terms are called conjugate surds.
Thus $(\sqrt{a}+\sqrt{b})$ and $(\sqrt{a}-\sqrt{b})$ are conjugate surds of each other.
The conjugate of $x+\sqrt{y}$ is $x-\sqrt{y}$.

The product of the conjugate surds $\sqrt{x}+\sqrt{y}$and $\sqrt{x}-\sqrt{y}$,

$$(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) = (\sqrt{x})^2-(\sqrt{y})^2 = x-y$$

is a rational quantity independent of any radical.
Similarly, the product of $x+m\sqrt{y}$ nd its conjugate $x-m\sqrt{y}$ has

$$(x+m\sqrt{y})(x-m\sqrt{y}) = (x)^2-(m\sqrt{y})^2$$
$$= x^2-m^2y$$

and have no radical. For example,

$$(4+\sqrt{3})(4-\sqrt{3}) = (4)^2-(\sqrt{3})^2 = 16-3 = 13$$

, which is a rational number.

EXERCISE 4.4

1. Rationalize the denominator of the following.

(i) $\frac{3}{4\sqrt{3}}$

Sol: $\frac{3}{4\sqrt{3}} = \frac{3}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$$= \frac{3\sqrt{3}}{4(\sqrt{3})^2}$$
$$= \frac{3\sqrt{3}}{4\times 3}$$
$$= \frac{\sqrt{3}}{4}$$

(ii) $\frac{14}{\sqrt{98}}$

Solution: $\frac{14}{\sqrt{98}} = \frac{14}{\sqrt{98}} \times \frac{\sqrt{98}}{\sqrt{98}}$

$$= \frac{14\sqrt{98}}{(\sqrt{98})^2}$$
$$= \frac{14\sqrt{98}}{98}$$
$$= \frac{\sqrt{98}}{7}$$

(iii) $\frac{6}{\sqrt{8}\sqrt{27}}$

Solution: $\frac{6}{\sqrt{8}\sqrt{27}} = \frac{6}{\sqrt{216}} = \frac{6}{\sqrt{216}} \times \frac{\sqrt{216}}{\sqrt{216}}$

$$= \frac{6\sqrt{216}}{(\sqrt{216})^2}$$
$$= \frac{6\sqrt{6\times 6\times 6}}{216}$$
$$= \frac{6\times 6\sqrt{6}}{216}$$
$$= \frac{\sqrt{6}}{6}$$

(iv) $\frac{1}{3+2\sqrt{5}}$

Solution: $\frac{1}{3+2\sqrt{5}} = \frac{1}{3+2\sqrt{5}} \times \frac{3-2\sqrt{5}}{3-2\sqrt{5}}$

$$= \frac{3-2\sqrt{5}}{(3)^2-(2\sqrt{5})^2}$$
$$= \frac{3-2\sqrt{5}}{9-(4\times 5)}$$
$$= \frac{3-2\sqrt{5}}{9-20}$$
$$= \frac{3-2\sqrt{5}}{-11}$$
$$= -\frac{1}{11}(3-2\sqrt{5})$$

(v) $\frac{15}{\sqrt{31}-4}$

Solution: $\frac{15}{\sqrt{31}-4} = \frac{15}{\sqrt{31}-4} \times \frac{\sqrt{31}+4}{\sqrt{31}+4}$

$$= \frac{15(\sqrt{31}+4)}{(\sqrt{31})^2-(4)^2}$$
$$= \frac{15(\sqrt{31}+4)}{31-16}$$
$$= \frac{15(\sqrt{31}+4)}{15}$$
$$= \sqrt{31}+4$$

(vi) $\frac{2}{\sqrt{5}-\sqrt{3}}$

Solution: $\frac{2}{\sqrt{5}-\sqrt{3}} = \frac{2}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

$$= \frac{2(\sqrt{5}+\sqrt{3})}{(\sqrt{5})^2-(\sqrt{3})^2}$$
$$= \frac{2(\sqrt{5}+\sqrt{3})}{5-3}$$
$$= \frac{2(\sqrt{5}+\sqrt{3})}{2}$$
$$= \sqrt{5}+\sqrt{3}$$

(vi) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

Solution: $\frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$
$$= \frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2-(1)^2}$$
$$= \frac{(\sqrt{3})^2+(1)^2-2(\sqrt{3})(1)}{3-1}$$
$$= \frac{3+1-2\sqrt{3}}{2}$$
$$= \frac{4-2\sqrt{3}}{2}$$
$$= \frac{2(2-\sqrt{3})}{2}$$
$$= 2-\sqrt{3}$$

(vi) $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

Sol: $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$
$$= \frac{(\sqrt{5}+\sqrt{3})^2}{(\sqrt{5})^2-(\sqrt{3})^2}$$
$$= \frac{(\sqrt{5})^2+(\sqrt{3})^2+2(\sqrt{5})(\sqrt{3})}{5-3}$$
$$= \frac{5+3+2\sqrt{15}}{2}$$
$$= \frac{8+2\sqrt{15}}{2}$$
$$= \frac{2(4+\sqrt{15})}{2}$$
$$= 4+\sqrt{15}$$

Q#2) Find the conjugate of $x+\sqrt{y}$.

(i). $3+\sqrt{7}$

Solution: Let $z = 3+\sqrt{7}$

Taking conjugate, we get
$$\bar{z} = \overline{3+\sqrt{7}}$$
$$\bar{z} = 3-\sqrt{7}$$

(ii). $4-\sqrt{5}$

Solution: Let $z = 4-\sqrt{5}$

Taking conjugate, we get
$$\bar{z} = \overline{4-\sqrt{5}}$$
$$\bar{z} = 4+\sqrt{5}$$

(iii). $2+\sqrt{3}$

Solution: Let $z = 2+\sqrt{3}$

Taking conjugate, we get
$$\bar{z} = \overline{2+\sqrt{3}}$$
$$\bar{z} = 2-\sqrt{3}$$

(iv). $2+\sqrt{5}$

Solution: Let $z = 2+\sqrt{5}$

Taking conjugate, we get
$$\bar{z} = \overline{2+\sqrt{5}}$$
$$\bar{z} = 2-\sqrt{5}$$

(v). $5+\sqrt{7}$

Solution: Let $z = 5+\sqrt{7}$

Taking conjugate, we get
$$\bar{z} = \overline{5+\sqrt{7}}$$
$$\bar{z} = 5-\sqrt{7}$$

(vi). $4-\sqrt{15}$

Solution: Let $z = 4-\sqrt{15}$

Taking conjugate, we get
$$\bar{z} = \overline{4-\sqrt{15}}$$
$$\bar{z} = 4+\sqrt{15}$$

(vii). $7-\sqrt{6}$

Solution: Let $z = 7-\sqrt{6}$

Taking conjugate, we get
$$\bar{z} = \overline{7-\sqrt{6}}$$
$$\bar{z} = 7+\sqrt{6}$$

(viii). $9+\sqrt{2}$

Solution: Let $z = 9+\sqrt{2}$

Taking conjugate, we get
$$\bar{z} = \overline{9+\sqrt{2}}$$
$$\bar{z} = 9-\sqrt{2}$$

Q#3)

(i). If $x = 2-\sqrt{3}$ find $\frac{1}{x}$

Solution: $x = 2-\sqrt{3}$

And $\frac{1}{x} = \frac{1}{2-\sqrt{3}} = \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$
$$= \frac{2+\sqrt{3}}{(2)^2-(\sqrt{3})^2}$$
$$= \frac{2+\sqrt{3}}{4-3}$$
$$= \frac{2+\sqrt{3}}{1}$$
$$= 2+\sqrt{3}$$

(ii). If $x = 4-\sqrt{17}$ find $\frac{1}{x}$

Solution: $x = 4-\sqrt{17}$

And $\frac{1}{x} = \frac{1}{4-\sqrt{17}} = \frac{1}{4-\sqrt{17}} \times \frac{4+\sqrt{17}}{4+\sqrt{17}}$
$$= \frac{4+\sqrt{17}}{(4)^2-(\sqrt{17})^2}$$
$$= \frac{4+\sqrt{17}}{16-17}$$
$$= \frac{4+\sqrt{17}}{-1}$$

$= -(4+\sqrt{17})$
$= -4-\sqrt{17}$

(iii). If $x = \sqrt{3}+2$ find $\frac{1}{x}$

Solution: $x = \sqrt{3}+2$

And $\frac{1}{x} = \frac{1}{\sqrt{3}+2} = \frac{1}{\sqrt{3}+2} \times \frac{\sqrt{3}-2}{\sqrt{3}-2}$

$$= \frac{\sqrt{3}-2}{(\sqrt{3})^2-(2)^2}$$
$$= \frac{\sqrt{3}-2}{3-4}$$
$$= \frac{\sqrt{3}-2}{-1}$$
$$= -(\sqrt{3}-2)$$
$$= -\sqrt{3}+2 = 2-\sqrt{3}$$

Q#4) Simplify

(vi) $\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}} + \frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$

Solution:

$$\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}} + \frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$$
$$= \left(\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}} \times \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right) + \left(\frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}\right)$$
$$= \frac{(1+\sqrt{2})(\sqrt{5}-\sqrt{3})}{(\sqrt{5})^2-(\sqrt{3})^2} + \frac{(1-\sqrt{2})(\sqrt{5}+\sqrt{3})}{(\sqrt{5})^2-(\sqrt{3})^2}$$
$$= \frac{[(1)(\sqrt{5})-(1)(\sqrt{3})+(\sqrt{2})(\sqrt{5})-(\sqrt{2})(\sqrt{3})]}{5-3} + \frac{[(1)(\sqrt{5})+(1)(\sqrt{3})-(\sqrt{2})(\sqrt{5})-(\sqrt{2})(\sqrt{3})]}{5-3}$$
$$= \frac{[\sqrt{5}-\sqrt{3}+\sqrt{10}-\sqrt{6}]}{2} + \frac{[\sqrt{5}+\sqrt{3}-\sqrt{10}-\sqrt{6}]}{2}$$
$$= \frac{(\sqrt{5}-\sqrt{3}+\sqrt{10}-\sqrt{6})+(\sqrt{5}+\sqrt{3}-\sqrt{10}-\sqrt{6})}{2}$$
$$= \frac{\sqrt{5}-\sqrt{3}+\sqrt{10}-\sqrt{6}+\sqrt{5}+\sqrt{3}-\sqrt{10}-\sqrt{6}}{2}$$
$$= \frac{2\sqrt{5}-2\sqrt{6}}{2}$$
$$= \frac{2(\sqrt{5}-\sqrt{6})}{2}$$
$$= \sqrt{5}-\sqrt{6}$$

(ii) $\frac{1}{2+\sqrt{3}} + \frac{2}{\sqrt{5}-\sqrt{3}} + \frac{1}{2+\sqrt{5}}$

Solution:

$$\frac{1}{2+\sqrt{3}} + \frac{2}{\sqrt{5}-\sqrt{3}} + \frac{1}{2+\sqrt{5}}$$
$$= \left(\frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}\right) + \left(\frac{2}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}\right) + \left(\frac{1}{2+\sqrt{5}} \times \frac{2-\sqrt{5}}{2-\sqrt{5}}\right)$$
$$= \frac{(2-\sqrt{3})}{(2)^2-(\sqrt{3})^2} + \frac{2(\sqrt{5}+\sqrt{3})}{(\sqrt{5})^2-(\sqrt{3})^2} + \frac{(2-\sqrt{5})}{(2)^2-(\sqrt{5})^2}$$
$$= \frac{(2-\sqrt{3})}{4-3} + \frac{2(\sqrt{5}+\sqrt{3})}{5-3} + \frac{(2-\sqrt{5})}{4-5}$$
$$= \frac{(2-\sqrt{3})}{1} + \frac{2(\sqrt{5}+\sqrt{3})}{2} + \frac{(2-\sqrt{5})}{-1}$$
$$= (2-\sqrt{3})+(\sqrt{5}+\sqrt{3})-(2-\sqrt{5})$$
$$= 2-\sqrt{3}+\sqrt{5}+\sqrt{3}-2+\sqrt{5}$$
$$= 2\sqrt{5}$$

(iii) $\frac{2}{\sqrt{5}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{2}} - \frac{3}{\sqrt{5}+\sqrt{2}}$

Solution:

$$\frac{2}{\sqrt{5}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{2}} - \frac{3}{\sqrt{5}+\sqrt{2}}$$
$$= \left(\frac{2}{\sqrt{5}+\sqrt{3}} \times \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}+\sqrt{3}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right) - \left(\frac{3}{\sqrt{5}+\sqrt{3}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}\right)$$
$$= \frac{2(\sqrt{5}-\sqrt{3})}{(\sqrt{5})^2-(\sqrt{3})^2} + \frac{(\sqrt{3}-\sqrt{2})}{(\sqrt{3})^2-(\sqrt{2})^2} - \frac{3(\sqrt{5}-\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2}$$
$$= \frac{2(\sqrt{5}-\sqrt{3})}{5-3} + \frac{(\sqrt{3}-\sqrt{2})}{3-2} - \frac{3(\sqrt{5}-\sqrt{2})}{5-2}$$
$$= \frac{2(\sqrt{5}-\sqrt{3})}{2} + \frac{(\sqrt{3}-\sqrt{2})}{1} - \frac{3(\sqrt{5}-\sqrt{2})}{3}$$
$$= (\sqrt{5}-\sqrt{3})+(\sqrt{3}-\sqrt{2})-(\sqrt{5}-\sqrt{2})$$
$$= \sqrt{5}-\sqrt{3}+\sqrt{3}-\sqrt{2}-\sqrt{5}+\sqrt{2}$$
$$= 0$$

Q#5)

(i). If $x = 2+\sqrt{3}$ find $x-\frac{1}{x}$ and $\left(x-\frac{1}{x}\right)^2$

Solution: $x = 2+\sqrt{3}$

And $\frac{1}{x} = \frac{1}{2+\sqrt{3}} = \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$

$$= \frac{2-\sqrt{3}}{(2)^2-(\sqrt{3})^2}$$
$$= \frac{2-\sqrt{3}}{4-3}$$
$$= \frac{2-\sqrt{3}}{1}$$
$$= 2-\sqrt{3}$$

Now, $x-\frac{1}{x} = (2+\sqrt{3})-(2-\sqrt{3})$

$$= 2+\sqrt{3}-2+\sqrt{3}$$
$$x-\frac{1}{x} = 2\sqrt{3}$$

Also, $\left(x-\frac{1}{x}\right)^2 = (2\sqrt{3})^2 = 4\times 3 = 12$

(i). If $x = \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ find $x+\frac{1}{x}$, $x^2+\frac{1}{x^2}$ and $x^3+\frac{1}{x^3}$

Solution: $x = \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$

$$= \frac{(\sqrt{5}-\sqrt{2})^2}{(\sqrt{5})^2-(\sqrt{2})^2}$$
$$= \frac{(\sqrt{5})^2+(\sqrt{2})^2-2(\sqrt{5})(\sqrt{2})}{5-2}$$
$$= \frac{5+2-2\sqrt{10}}{3}$$
$$= \frac{1}{3}(7-2\sqrt{10})$$

And $\frac{1}{x} = \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}} \times \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}$

$$= \frac{(\sqrt{5}+\sqrt{2})^2}{(\sqrt{5})^2-(\sqrt{2})^2}$$
$$= \frac{(\sqrt{5})^2+(\sqrt{2})^2+2(\sqrt{5})(\sqrt{2})}{5-2}$$
$$= \frac{5+2+2\sqrt{10}}{3}$$
$$= \frac{1}{3}(7+2\sqrt{10})$$

Now, $x+\frac{1}{x} = \frac{1}{3}(7-2\sqrt{10})+\frac{1}{3}(7+2\sqrt{10})$

$$= \frac{1}{3}(7-2\sqrt{10}+7+2\sqrt{10})$$
$$x+\frac{1}{x} = \frac{14}{3}$$

Using identity

$$\left(x+\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}+2$$

Putting values

$$\left(\frac{14}{3}\right)^2 = x^2+\frac{1}{x^2}+2$$
$$\frac{196}{9} = x^2+\frac{1}{x^2}+2$$
$$x^2+\frac{1}{x^2} = \frac{196}{9}-2$$
$$x^2+\frac{1}{x^2} = \frac{196-18}{9}$$
$$x^2+\frac{1}{x^2} = \frac{178}{9}$$

Also using the identity

$$\left(x+\frac{1}{x}\right)^3 = x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)$$

Putting values

$$\left(\frac{14}{3}\right)^3 = x^3+\frac{1}{x^3}+3\left(\frac{14}{3}\right)$$

$\frac{2744}{27} = x^3+\frac{1}{x^3}+14$

$$x^3+\frac{1}{x^3} = \frac{2744}{27}-14$$
$$x^3+\frac{1}{x^3} = \frac{2366}{27}$$

Q#6) Determine the rational numbers a and b if,

$\frac{\sqrt{3}-1}{\sqrt{3}+1}+\frac{\sqrt{3}+1}{\sqrt{3}-1} = a+\sqrt{3}b$

Solution:

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}+\frac{\sqrt{3}+1}{\sqrt{3}-1} = a+\sqrt{3}b$$

$$\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}\right)+\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$$
$$= a+\sqrt{3}b$$
$$\left(\frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2-(1)^2}\right)+\left(\frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2-(1)^2}\right) = a+\sqrt{3}b$$

$$\frac{(\sqrt{3})^2+(1)^2-2(\sqrt{3})(1)}{3-1}$$
$$+\frac{(\sqrt{3})^2+(1)^2+2(\sqrt{3})(1)}{3-1}$$
$$= a+\sqrt{3}b$$
$$\frac{3+1+2\sqrt{3}}{2}+\frac{3+1-2\sqrt{3}}{2} = a+\sqrt{3}b$$
$$\frac{4+2\sqrt{3}+4-2\sqrt{3}}{2} = a+\sqrt{3}b$$
$$\frac{8}{2} = a+\sqrt{3}b$$
$$4+0\sqrt{3} = a+\sqrt{3}b$$

On comparing, we get

$a = 4$ and $b = 0$

Complied by Shumaila Amin